Conservation laws, bright matter wave solitons and modulational instability of nonlinear Schrödinger equation with time-dependent nonlinearity *

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Abstract: In this paper, we consider a general form of nonlinear Schrödinger equation with time-dependent nonlinearity. Based on the linear eigenvalue problem, the complete integrability of such nonlinear Schrödinger equation is identified by admitting an infinite number of conservation laws. Using the Darboux transformation method, we obtain some explicit bright multi-soliton solutions in a recursive manner. The propagation characteristic of solitons and their interactions under the periodic plane wave background are discussed. Finally, the modulational instability of solutions is analyzed in the presence of small perturbation.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

It is known that every atom in a Bose-Einstein Condensates (BECs) moves in an effective mean field due to the other atoms and the mean field equation of motion governing the evolution of the macroscopic wave function of the Bose-Einstein condensate is the so-called time-dependent Gross-Pitaevskii (GP) equation [1]

$$i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t}=\left[-\frac{\hbar^2\nabla^2}{2m}+V_{ext}(\vec{r})+g|\Psi(\vec{r},t)|^2\right]\Psi(\vec{r},t),\tag{1.1}$$

where Ψ is the BEC order parameter, V_{ext} is the external trapping potential and the coefficient $g=4\pi\hbar^2a/m$ characterizes the effective interatomic interactions in the BEC through the s-wave scattering

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length a. For a parabolic potential and time dependent scattering lengths $V_{ext} = -\frac{\epsilon^2 x^2}{4}\hbar$ with $a(t) = -\frac{2\pi\hbar a}{m}$, $\hbar = 2m$, the above GP equation in one dimension takes the following nonlinear Schrödinger equation with time-dependent nonlinearity [2-5]

$$i\frac{\partial\psi(x,t)}{\partial t} + \frac{\partial^2\psi(x,t)}{\partial x^2} + 2a(t)|\psi(x,t)|^2\psi(x,t) + \frac{1}{4}\epsilon^2 x^2\psi(x,t) = 0. \tag{1.2}$$

Here the Feshbach-managed nonlinear coefficient $a(t)$ can be redefined as $a(t) = |a_s(t)|/a_B = g_0 \exp(\epsilon t)$ (a_B is the Bohr radius)[6], which is also called the time dependent scattering length. In Eq.(1.2), time t and coordinate x are measured in units $2/\omega_\perp$ and $a_\perp$, where $a_\perp = (\hbar/m\omega_\perp)^{1/2}$ and $a_0 = (\hbar/m\omega_0)^{1/2}$ are linear oscillator lengths in the transverse and cigar-axis directions, respectively. $\omega_\perp$ and ω_0 are corresponding harmonic oscillator frequencies, m is the atomic mass, and the parameter $\epsilon = 2|\omega_0|/\omega_\perp \ll 1$.

Investigation of the behaviour of Bose-Einstein Condensates (BECs) requires solving an inhomogeneous nonlinear Schrödinger equation known as the GP equation [1]. Eventhough numerical solutions of GP equation are available [7, 8], construction of analytic solutions will offer more insight into the domain of BECs opening the doors for developing concrete applications of BECs in future. As we well know, it is significantly important in mathematical physics to search for exact solutions to equation (1.2). Exact solutions play a vital role in understanding various qualitative and quantitative features of nonlinear phenomena. It is well known that searching for soliton solutions of the nonlinear evolution equations is one of the most important topics in soliton theory. Darboux transformation (DT) [9, 10] has been proven to be one of the most fruitful algorithmic procedures to obtain exact solutions of the nonlinear evolution equations.

The main aim of the present paper is to construct some infinite number of conservation laws, explicit bright multi-soliton solutions by using DT method, and investigate the modulational instability of solutions of a general form of nonlinear Schrödinger equation with time-dependent nonlinearity (1.2). In this paper, on the basis of the Lax pair associated with Eq.(1.2), we will derive an infinite number of conservation laws to identify its complete integrability. Furthermore, we will apply the Darboux transformation method to this integrable model and give the general procedure to recursively generate the bright N-soliton solutions from an initial trivial solution. Moreover, we will discuss the propagation characteristic and interactions of solitons under periodic plane wave background and analyze the linear stability of the nonlinear plane waves.

The paper will be organized as follows: In Section 2, based on the linear eigenvalue problem associated with Eq.(1.2) and an auxiliary 1D NLSE (2.2), we obtain a series of conservation laws. Then the integrability is identified by admitting an infinite number of conservation laws. In Section 3, we will apply the Darboux transformation method to this integrable model and give the general procedure to recursively generate the bright N-soliton solutions from an initial trivial solution. The propagation characteristic of solitons and their interactions under the periodic plane wave background are discussed in Section 4. In Section 5, we analyze the modulational instability of the nonlinear plane waves. Finally, some conclusions and discussions are provided.

2. Lax pair and infinite conservation laws

In this section, using the linear eigenvalue problem associated with 1D NLSE (1.2) and an auxiliary 1D NLSE (2.2), we construct an infinite number of conservation laws for the 1D NLSE (1.2). Then the integrability is identified by admitting an infinite number of conservation laws.

Under the following transformation

$$\psi(x,t) = u(x,t)e^{-i\frac{\epsilon x^2}{4}-\frac{\epsilon t}{2}} \tag{2.1}$$

to Eq.(1.2), we can obtain the following new 1D NLSE

$$i\frac{\partial}{\partial t}u(x,t) + \frac{\partial^2}{\partial x^2}u(x,t) - i\epsilon x\frac{\partial}{\partial x}u(x,t) + 2a(t)|u(x,t)|^2u(x,t)e^{-\epsilon t} - i\epsilon u(x,t) = 0. \tag{2.2}$$

By virtue of the Ablowitz-Kaup-Newell-Segur scheme [11, 12], the Lax pair associated with equation (2.2) can be derived as

$$\Phi_x = U\Phi, \quad \Phi_t = V\Phi, \tag{2.3}$$

where $\Phi = (\phi_1, \phi_2)^T$ (the superscript T denotes the vector transpose) is the vector eigenfunction[2], and the matrices U and V have the following forms

$$U = \lambda J + P, \tag{2.4a}$$

$$V = 2i\lambda^2 J + \epsilon\lambda xJP + 2i\epsilon P + Q, \tag{2.4b}$$

with

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & \sqrt{g_0}u(x,t) \\ -\sqrt{g_0}u(x,t)^* & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} ig_0|u(x,t)|^2 & -\sqrt{g_0}\epsilon xu(x,t) + i\sqrt{g_0}u(x,t)_x \\ \sqrt{g_0}\epsilon xu(x,t)^* + i\sqrt{g_0}u(x,t)^*_x & -ig_0|u(x,t)|^2 \end{pmatrix},$$

where $g_0 = a(t)\exp(-\epsilon t)$ is an arbitrary function. Hereafter the asterisk stands for the complex conjugate. From the compatibility condition $U_t - V_x + [U, V] = 0$, one can derive Eq.(2.2) and Eq.(1.2) in the case of $\psi(x,t) = u(x,t)e^{-i\frac{\epsilon x^2}{4}-\frac{\epsilon t}{2}}$, respectively.

Using Lax pair (2.3) and Refs.[13, 14], we can further derive an infinite number of conservation laws of Eqs.(1.2) and (2.2). By introducing the quantity $f(x,t) = \sqrt{g_0}u(x,t)\frac{\phi_2}{\phi_1}$, the linear equations (2.3) can be transformed into the following Ricatti equation

$$f(x,t)_x + i\frac{u(x,t)_{xx}}{u(x,t)}f(x,t) + \alpha(x,t)\frac{u(x,t)_x}{u(x,t)}f(x,t) - 2ig_0f(x,t)^2 - 2ig_0u(x,t)u(x,t)^* + \beta(x,t)f(x,t) = 0, \tag{2.5}$$

with $\alpha(x,t) = 2i\epsilon - 2i + \epsilon\lambda x - \epsilon x + i\lambda$ and $\beta(x,t) = \epsilon\lambda - \epsilon + \epsilon\lambda^2 x + 2i\epsilon\lambda - \epsilon\lambda x - 4i\lambda^2$. Substituting $f(x,t) = \sum_{n=1}^{\infty}\frac{f_n}{\beta(x,t)^n}$ into Ricatti equation (2.5) and equating the like powers of $\beta(x,t)$ to zero, we have a recursion formula

$$f_1 = 2ig_0uu^*, \quad f_{n+1} = 2ig_0\sum_{i=1}^{n}f_if_{n-i} - f_{nx} - i\frac{u_{xx}}{u}f_n - \alpha\frac{u_x}{u}f_n = 0, \quad (n = 2, 3, \cdots), \tag{2.6}$$

where f_n $(n = 1, 2, \cdots)$ are the functions to be determined. By virtue of the compatibility condition $(\ln \phi_1)_{xt} = (\ln \phi_1)_{tx}$, we obtain the following conservation form

$$i\frac{\partial}{\partial t}\rho_k(x,t) + \frac{\partial}{\partial x}\Im_k(x,t) = 0, \tag{2.7}$$

where $\rho_k(x,t)$ and $\Im_k(x,t)$ $(k = 1, 2, \cdots)$ are called conserved densities and conserved fluxes, respectively. The first three significant physical conservation laws are presented as

$$\begin{aligned}
\rho_1(x,t) &= 2ig_0|u|^2, \quad \rho_2(x,t) = -2ig_0(1+\alpha)u_x u^* - 2ig_0 u u_x^* + 2g_0 u_{xx} u^*,\\
\rho_3(x,t) &= -8ig_0^3|u|^4 - 2g_0 u_{xx}u_x^* + 2i(1+\alpha)u_{xx}u^* + 2i\epsilon g_0(\lambda-1)u_x u^* + 2i(1+3\alpha)g_0 u_x u_x^*\\
&\quad - \frac{1}{u}\left[2ig_0 u_{xx}^2 u^* - 2i\alpha(1+\alpha)g_0 u_x^2 u^* + 2(1+2\alpha)g_0 u_x u_{xx} u^*\right],\\
\Im_1(x,t) &= ig_0|u|^2 - 2g_0\sqrt{g_0}u_x u^* + 2i(\epsilon\lambda x + 2i\epsilon - \epsilon x)g_0\sqrt{g_0}uu^*,\\
\Im_2(x,t) &= ig_0|u|^2 + 2g_0\sqrt{g_0}u_x u_x^* + 2ig_0\sqrt{g_0}\frac{u_x u_{xx}u^*}{u} + 2(1+\alpha)g_0\sqrt{g_0}\frac{u_x^2 u^*}{u} - 2ig_0\sqrt{g_0}(\epsilon\lambda x + 2i\epsilon\\
&\quad - \epsilon x)\left[uu_x^* + iu_{xx}u^* + (1+\alpha)u_x u^*\right],\\
\Im_3(x,t) &= ig_0|u|^2 + \sqrt{g_0}\left(\epsilon\lambda x + 2i\epsilon - \epsilon x + i\frac{u_x}{u}\right)\\
&\quad \times\left[-8ig_0^3|u|^4 - 2g_0 u_{xx}u_x^* + 2i(1+\alpha)u_{xx}u^* + 2i\epsilon(\lambda-1)g_0 u_x u^* + 2i(1+3\alpha)g_0 u_x u_x^*\right]\\
&\quad - \frac{\sqrt{g_0}}{u}\left(\epsilon\lambda x + 2i\epsilon - \epsilon x + i\frac{u_x}{u}\right)\left[2ig_0 u_{xx}^2 u^* - 2i\alpha(1+\alpha)g_0 u_x^2 u^* + 2(1+2\alpha)g_0 u_x u_{xx}u^*\right].
\end{aligned}$$

The conserved quantities $\mathscr{J} = 2ig_0\int_{-\infty}^{+\infty}|u|^2 dx$, $\mathscr{H} = -2ig_0\int_{-\infty}^{+\infty}(1+\alpha)u_x u^* + uu_x^* + iu_{xx}u^* dx$ and $\mathscr{K} = 2ig_0\int_{-\infty}^{\infty} -4g_0^2|u|^4 + iu_{xx}u_x^* + \frac{1}{g_0}(1+\alpha)u_{xx}u^* + \epsilon(\lambda-1)u_x u^* + (1+3\alpha)u_x u_x^* - \frac{1}{u}\left[u_{xx}^2 u^* - \alpha(1+\alpha)u_x^2 u^* - i(1+2\alpha)u_x u_{xx}u^*\right]dx$ represent the energy, momentum and Hamiltonian, respectively.

3. Darboux transformation and bright soliton solutions

In this section, the Darboux transformation method is applied to this integrable model and the general procedure is presented to recursively generate the bright N-soliton solutions from an initial trivial solution. Based on Lax pair (2.3), the use of the Darboux transformation to construct the bright soliton solution of nonlinear partial differential equations is an optimum choice among various methods in soliton theory [15-26]. Owing to its purely algebraic algorithm, with the symbolic computation, the analytical N-soliton solution can be generated through successive application of the Darboux transformation [31,32].

3.1 Darboux transformation

The Darboux transformation is actually a gauge transformation $\Phi[1] = T\Phi$ of the spectral problem (2.3) by considering the following gauge transformation

$$\Phi[1] = (\lambda I - S)\Phi \;\text{ with }\; S = H\Lambda H^{-1}, \;\; \Lambda = \mathrm{diag}(\lambda_1, \lambda_1^*), \tag{3.1}$$

where H is a nonsingular matrix. It is required that $\Phi[1]$ solves the same spectral problems (2.3)

$$\Phi[1]_x = U[1]\Phi[1] \;\text{ and }\; \Phi[1]_t = V[1]\Phi[1], \tag{3.2}$$

with $U[1] = \lambda J + P[1], V[1] = 2i\lambda^2 J + \epsilon\lambda xJP[1] + 2i\epsilon P[1] + Q[1]$ and

$$P[1] = \begin{pmatrix} 0 & \sqrt{g_0}u[1] \\ -\sqrt{g_0}u[1]^* & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$Q[1] = \begin{pmatrix} ig_0|u[1]|^2 & -\sqrt{g_0}\epsilon xu[1] + i\sqrt{g_0}u[1]_x \\ \sqrt{g_0}\epsilon xu[1]^* + i\sqrt{g_0}u[1]^*_x & -ig_0|u[1]|^2 \end{pmatrix}, \tag{3.3}$$

where $u[1] = \psi[1]e^{i\frac{\epsilon x^2}{4}+\frac{\epsilon t}{2}}$. Substituting Eq. (3.1) into Eqs. (3.2), we have the following relationship

$$P[1] = P + [J, S], \quad S_x + [S, JS + P] = 0. \tag{3.4}$$

Now we discuss a concrete transformation. It is easy to verify that if $\Phi = (\phi_1, \phi_2)^T$ is an eigenfunction of Eqs. (2.3) with eigenvalue $\lambda = \lambda_1$, then $(\phi_2, -\phi_1^*)^T$ is also an eigenfunction of Eqs. (2.3) with eigenvalue $\lambda = \lambda_1^*$. Thus we take the matrix H in the form

$$H = \begin{pmatrix} \phi_1 & \phi_2^* \\ \phi_2 & -\phi_1^* \end{pmatrix}. \tag{3.5}$$

Therefore, by means of Eqs. (3.4) and (3.5), the onceiterated new potential of Eqs.(1.2) and (2.2) are given by

$$\psi[1] = u[1]e^{-i\frac{\epsilon x^2}{4}-\frac{\epsilon t}{2}},$$
$$u[1] = u + 2\frac{(\lambda_1 - \lambda_1^*)\phi_1\phi_2^*}{\phi_1\phi_1^* + \phi_2\phi_2^*}. \tag{3.6}$$

It is straightforward to verify that the Darboux transformations (3.1) and (3.6) can simultaneously preserve the form of linear eigenvalue problem (2.3).

According to the above computation, we have the following propositions.

Proposition 3.1. *When $u(x,t)$ is given, let $(\phi_1, \phi_2)^T$ be the solution of Eqs.*(2.3) *with $\lambda = \lambda_k$, then by using of the Darboux matrix T* (3.1) *and Darboux transformation* (3.6)*, we have*

$$U[1] = (T_x + TU)T^{-1}, \quad \textit{with} \quad T = \lambda I - S. \tag{3.7}$$

We can obtain the same proposition about the auxiliary spectral problem.

Proposition 3.2. *When $u(x,t)$ is given, let $(\phi_1, \phi_2)^T$ be the solution of Eqs.*(2.3) *with $\lambda = \lambda_k$, then by using of the Darboux matrix T* (3.1) *and Darboux transformation* (3.6)*, we have*

$$V[1] = (T_t + TV)T^{-1} \; \textit{with} \; T = \lambda I - S. \tag{3.8}$$

Proof. Substituting Eqs. (3.1), (3.2) and (3.6) into Eq. (3.8) by a direct calculation, we can obtain the conclusion. The proof is completed. □

Propositions 3.1 and 3.2 show that the transformation (3.1) and (3.6) change the Lax pair (2.3) into another Lax pair of the type (3.2) with $U[1]$ and $V[1]$ having the same form as U and V, respectively. Therefore both of the Lax pairs lead to the same equation (2.2), so Eqs. (3.1) and (3.6) are the

Darboux transformation of Eq.(2.2). From propositions 3.1 and 3.2, we have following theorem.

Theorem 3.3. *Assuming that* $\psi, u, (\phi[1,\lambda_1]_1, \phi[1,\lambda_1]_2)^T, (\phi[2,\lambda_2]_1, \phi[2,\lambda_2]_2)^T, \cdots, (\phi[N,\lambda_N]_1, \phi[N,\lambda_N]_2)^T$ *be the solution of the 1D NLSE* (2.2) *and N linearly independent solutions of the linear eigenvalue problem* (2.3)*, respectively, and after iterating the Darboux transformation* (3.1) *and* (3.6) *N times analogous to the above procedure, we can further obtain the Nth-iterated potential transformation as*

$$\begin{aligned}
&\psi[N] = u[N]e^{-i\frac{\epsilon x^2}{4}-\frac{\epsilon t}{2}},\\
&u[N] = u + 2\sum_{i=1}^{N}\frac{(\lambda_i-\lambda_i^*)\phi[i,\lambda_i]_1\phi[i,\lambda_i]_2^*}{\phi[i,\lambda_i]_1\phi[i,\lambda_i]_1^* + \phi[i,\lambda_i]_2\phi[i,\lambda_i]_2^*},
\end{aligned} \tag{3.9}$$

with

$$\begin{aligned}
&\phi[i+1,\lambda_{i+1}]_j = (\lambda_{i+1}-\lambda_i^*)\phi[i,\lambda_{i+1}]_j - \frac{\mathscr{A}_i}{\mathscr{B}_i}(\lambda_i-\lambda_i^*)\phi[i,\lambda_i]_j,\\
&\mathscr{A}_i = \phi[i,\lambda_i]_1^*\phi[i,\lambda_{i+1}]_1 + \phi[i,\lambda_i]_2^*\phi[i,\lambda_{i+1}]_2,\\
&\mathscr{B}_i = \phi[i,\lambda_{i+1}]_1\phi[i,\lambda_{i+1}]_1^* + \phi[i,\lambda_{i+1}]_2\phi[i,\lambda_{i+1}]_2^*,\quad (i = 1,2,\cdots,N-1,\ j = 1,2),
\end{aligned} \tag{3.10}$$

where $(\phi[k,\lambda_k]_1, \phi[k,\lambda_k]_2^*)^T$ $(k = 1,2,\cdots,N)$ *is the eigenfunction of Eqs.* (2.3) *with the eigenvalue* $\lambda = \lambda_k$ *and potential* $u[k-1]$.

Proof. (By induction) The case $N = 1$ follows from Darboux transformation (3.1) and (3.6). Now assume that equation (3.9) holds for $N = n$. Then

$$\begin{aligned}
&\psi[n] = u[n]e^{-i\frac{\epsilon x^2}{4}-\frac{\epsilon t}{2}},\\
&u[n] = u + 2\sum_{i=1}^{n}\frac{(\lambda_i-\lambda_i^*)\phi[i,\lambda_i]_1\phi[i,\lambda_i]_2^*}{\phi[i,\lambda_i]_1\phi[i,\lambda_i]_1^* + \phi[i,\lambda_i]_2\phi[i,\lambda_i]_2^*},
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
&\phi[i+1,\lambda_{i+1}]_j = (\lambda_{i+1}-\lambda_i^*)\phi[i,\lambda_{i+1}]_j - \frac{\mathscr{A}_i}{\mathscr{B}_i}(\lambda_i-\lambda_i^*)\phi[i,\lambda_i]_j,\\
&\mathscr{A}_i = \phi[i,\lambda_i]_1^*\phi[i,\lambda_{i+1}]_1 + \phi[i,\lambda_i]_2^*\phi[i,\lambda_{i+1}]_2,\\
&\mathscr{B}_i = \phi[i,\lambda_{i+1}]_1\phi[i,\lambda_{i+1}]_1^* + \phi[i,\lambda_{i+1}]_2\phi[i,\lambda_{i+1}]_2^*,\quad (i = 1,2,\cdots,n-1,\ j = 1,2).
\end{aligned} \tag{3.12}$$

In case of $N = n+1$, using Darboux transformation (3.6), one obtains

$$\begin{aligned}
u[n+1] =&u[n] + 2\frac{(\lambda_{n+1}-\lambda_{n+1}^*)\phi[n+1,\lambda_{n+1}]_1\phi[n+1,\lambda_{n+1}]_2^*}{\phi[n+1,\lambda_{n+1}]_1\phi[n+1,\lambda_{n+1}]_1^* + \phi[n+1,\lambda_{n+1}]_2\phi[n+1,\lambda_{n+1}]_2^*},\\
=&u + 2\sum_{i=1}^{n+1}\frac{(\lambda_i-\lambda_i^*)\phi[i,\lambda_i]_1\phi[i,\lambda_i]_2^*}{\phi[i,\lambda_i]_1\phi[i,\lambda_i]_1^* + \phi[i,\lambda_i]_2\phi[i,\lambda_i]_2^*}.\ \text{(with } Eq.\text{(3.11))}
\end{aligned} \tag{3.13}$$

Based on the first equation of Darboux transformation (3.6), it is easy to obtain

$$\psi[n+1] = u[n+1]e^{-i\frac{\epsilon x^2}{4}-\frac{\epsilon t}{2}}. \tag{3.14}$$

From Eqs.(3.13) and (3.14), the conclusions of Eqs.(3.9) and (3.10) are hold for $N = n+1$. This completes the proof. □

3.2 Bright matter wave soliton solution

Bright soliton here is defined as the emergence of a positive pulse. Bright soliton solutions have been investigated by original Refs.[11, 27, 28]. In the following, the Darboux transformation is applied to construct the explicit bright soliton solutions of Eq.(1.2). Using the trivial solution $\psi = 0$ (u=0), we solve the linear equations (2.3) with $\lambda = \lambda_1 = \frac{1}{2}(\mu_1 + i\nu_1)$ and obtain the eigenfunction

$$\phi_1 = e^{\frac{1}{2}(\wp_1 + i\vartheta_1)}, \ \ \phi_2 = e^{-\frac{1}{2}(\wp_1 + i\vartheta_1)}, \tag{3.15}$$

with

$$\wp_1 = \mu_1 x - 2\mu_1\nu_1 t\xi_1, \ \ \vartheta_1 = \nu_1 x + \left(\mu_1^2 - \nu_1^2\right) t + \eta_1, \tag{3.16}$$

where ξ_1 and η_1 are arbitrary constants.

The bright one-soliton solution for Eq.(1.2) are obtained as

$$\begin{aligned} \psi(x,t) &= u(x,t) e^{-i\frac{\epsilon x^2}{4} - \frac{\epsilon t}{2}}, \\ u(x,t) &= i\nu_1 e^{i\vartheta_1} \mathrm{sech}\wp_1. \end{aligned} \tag{3.17}$$

by substitution of the above results into formula (3.6). It implies that the imaginary part ν_1 of the eigenvalue λ_1 determines the amplitude of the solitons $\psi(x,t)$ and $u(x,t)$, while the velocity of solitons are related to both real and imaginary parts of the eigenvalue λ_1.

From above, we have the following propositions.

Proposition 3.4. *Assuming that* $(\phi[1,\lambda_1]_1, \phi[1,\lambda_1]_2)^T$ *and* $(\phi[2,\lambda_2]_1, \phi[2,\lambda_2]_2)^T$ *be two linearly independent solutions of the linear eigenvalue problem* (2.3) *corresponding to two different eigenvalues* $\lambda_1 = \frac{1}{2}(\mu_1 + i\nu_1)$ *and* $\lambda_2 = \frac{1}{2}(\mu_2 + i\nu_2)$*, respectively, we obtain the bright two-soliton solutions of Eq.*(1.2) *and Eq.*(2.2) *from formula* (3.9) *with* $N = 2$

$$\begin{aligned} \psi[2] &= u[2] e^{-i\frac{\epsilon x^2}{4} - \frac{\epsilon t}{2}}, \\ u[2] &= i\frac{\mathscr{U}_1 e^{i\vartheta_2}\cosh\wp_1 + \mathscr{U}_2 e^{i\vartheta_1}\cosh\wp_2 + \mathscr{U}_3\left(e^{i\vartheta_1}\sinh\wp_2 - e^{i\vartheta_2}\sinh\wp_1\right)}{\mathscr{V}_1\cosh(\wp_1 + \wp_2) + \mathscr{V}_2\cosh(\wp_1 - \wp_2) - 2\nu_1\nu_2\cosh(\vartheta_2 - \vartheta_1)}, \end{aligned} \tag{3.18}$$

with

$$\begin{aligned} &\wp_i = \mu_i x - 2\mu_i\nu_i t\xi_i, \ \ \vartheta_i = \nu_i x + \left(\mu_i^2 - \nu_i^2\right) t + \eta_i, \\ &\mathscr{U}_1 = \nu_2\left[(\mu_1 - \mu_2)^2 - \nu_1^2 + \nu_2^2\right], \ \ \mathscr{U}_2 = \nu_1\left[(\mu_1 - \mu_2)^2 + \nu_1^2 - \nu_2^2\right], \\ &\mathscr{U}_3 = -2i\nu_1\nu_2(\mu_1 - \mu_2), \ \ \mathscr{V}_1 = \frac{1}{2}(\mu_1 - \mu_2)^2 + \frac{1}{2}(\nu_1 - \nu_2)^2, \\ &\mathscr{V}_2 = \frac{1}{2}(\mu_1 - \mu_2)^2 + \frac{1}{2}(\nu_1 + \nu_2)^2, \ \ (i = 1, 2), \end{aligned} \tag{3.19}$$

where ξ_i, η_i *are all complex constants and the parameters* $\mu_i, \nu_i \neq 0 (i = 1, 2)$.

Proof. It is straightforward to prove this proposition by using the initial values (3.15), (3.16), and Theorem 3.3 for $u = 0$, $N = 2$. □

Proposition 3.5. *Assuming that* $(\phi[1,\lambda_1]_1, \phi[1,\lambda_1]_2)^T, (\phi[2,\lambda_2]_1, \phi[2,\lambda_2]_2)^T, \cdots, (\phi[N,\lambda_N]_1, \phi[N,\lambda_N]_2)^T$ *be N linearly independent solutions of the linear eigenvalue problem* (2.3) *corresponding to N different eigenvalues* $\lambda_1 = \frac{1}{2}(\mu_1 + i\nu_1), \lambda_2 = \frac{1}{2}(\mu_2 + i\nu_2), \cdots, \lambda_N = \frac{1}{2}(\mu_N + i\nu_N)$*, respectively, we obtain the*

bright N-soliton solutions of Eqs.(1.2) *and* (2.2) *from formula* (3.9) *with N*

$$
\begin{aligned}
\psi[N] &= u[N]e^{-i\frac{\epsilon x^2}{4}-\frac{\epsilon t}{2}},\\
u[N] &= 2\sum_{i=1}^{N}\frac{(\lambda_i-\lambda_i^*)\phi[i,\lambda_i]_1\phi[i,\lambda_i]_2^*}{\phi[i,\lambda_i]_1\phi[i,\lambda_i]_1^*+\phi[i,\lambda_i]_2\phi[i,\lambda_i]_2^*},
\end{aligned}
\tag{3.20}
$$

with

$$
\wp_i=\mu_i x-2\mu_i\nu_i t\xi_i,\quad \vartheta_i=\nu_i x+\left(\mu_i^2-\nu_i^2\right)t+\eta_i,\qquad (i=1,2,\cdots,N),
\tag{3.21}
$$

where ξ_i,η_i *are all real constants and the parameters* $\mu_i,\nu_i\neq 0 (i=1,2,\cdots,N)$. $(\phi[k,\lambda_k]_1,\phi[k,\lambda_k]_2^*)^T$ *satisfies Eq.*(3.10) *and is the eigenfunction of Eqs.* (2.3) *with the eigenvalue* $\lambda=\lambda_k$ *and potential* $u[k]$ $(k=1,2,\cdots,N)$.

Proof. It is straightforward to prove this proposition by using the initial values (3.15), (3.16), and Theorem 3.3 for $u=0$. □

The graphs of the bright one-soliton periodic wave solution (3.17) and two-soliton periodic wave solution (3.18) are plotted in Fig. 1 and Fig. 2, respectively.

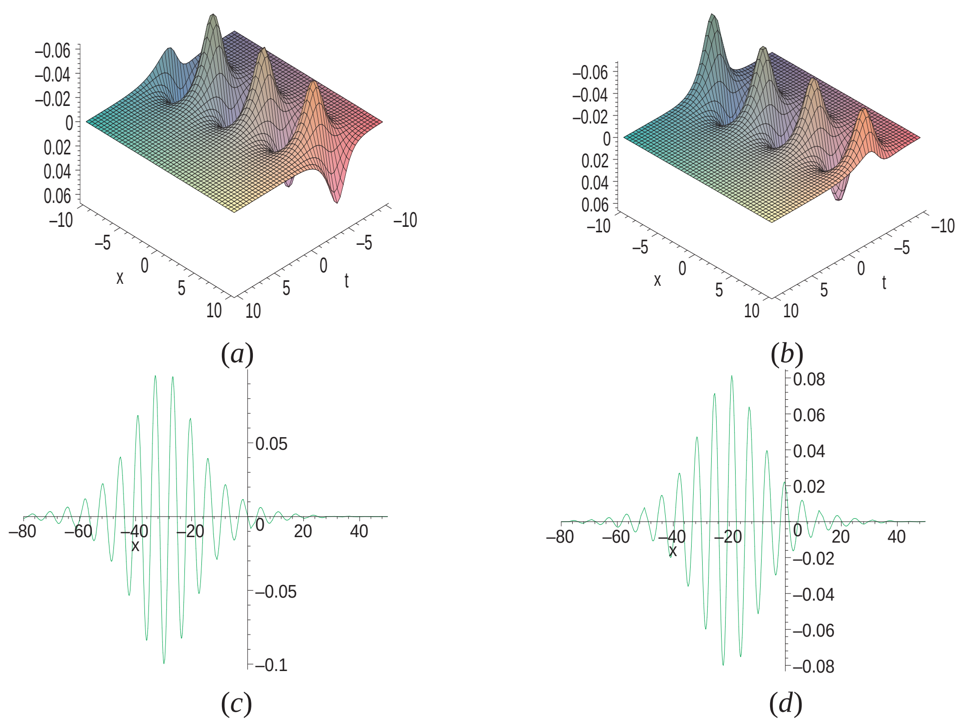


Fig. 1. (Color online) A symmetric bright one-soliton solution $|\psi(x,t)|^2$ of Eq.(1.2) with parameters: $\mu_1=1$, $\nu_1=0.1$, $\xi_1=1+i$, $\eta_1=3$ and $\epsilon=0.01$. This figure shows that the symmetric bright one-soliton periodic wave is spatially periodic in two directions, but it need not be periodic in either the x or t directions. (a) Perspective view for the real part of wave. (b) Perspective view of the imaginary part wave. (c) Wave propagation pattern of the real part wave along the x axis. (d) Wave propagation pattern of the imaginary part wave along the x axis.

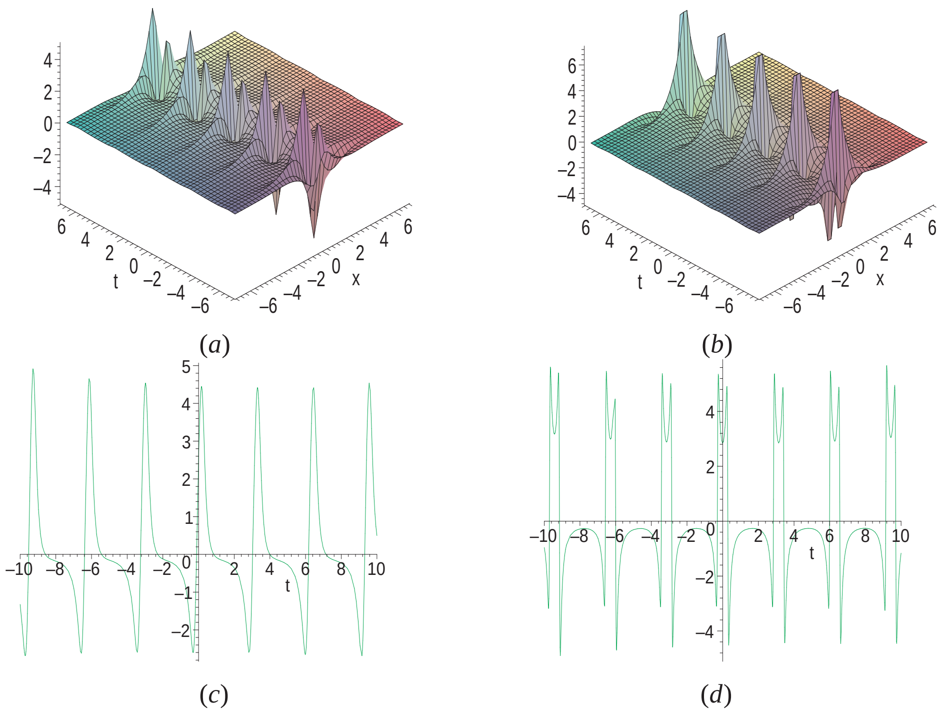


Fig. 2. (Color online) A symmetric bright two-soliton solution $|\psi(x,t)|^2$ of Eq.(1.2) with parameters: $\mu_1 = 0.1$, $\nu_1 = 0.1$, $\xi_1 = i$, $\eta_1 = 0.1i$, $\mu_2 = 1$, $\nu_1 = 1$, $\xi_2 = i$, $\eta_2 = 0.1i$, and $\epsilon = 0.01$. This figure shows that the symmetric bright two-soliton periodic wave is spatially periodic in two directions, but it is periodic in either the x or t directions. (a) Perspective view for the real part of wave. (b) Perspective view of the imaginary part wave. (c) Wave propagation pattern of the real part wave along the t axis. (d) Wave propagation pattern of the imaginary part wave along the t axis.

4. Bright soliton interactions on the periodic background

In this section, we investigate some bright soliton interactions on the periodic background by using Eqs.(3.6) and (3.9). The simple exact solution of Eq.(1.2) is the plane wave

$$\begin{aligned}\psi(x,t)_0 &= u(x,t)_0 e^{-i\frac{\epsilon x^2}{4} - \frac{\epsilon t}{2}},\\ u(x,t)_0 &= \gamma e^{i(kx+\omega t)},\end{aligned} \tag{4.1}$$

where γ and k are real constants, and the frequency ω solves the nonlinear dispersion relation

$$\omega = -k^2 + \epsilon k + 2\gamma^2 - i\epsilon + \frac{\epsilon^2}{4}. \tag{4.2}$$

Considering solution (4.1) as the initial seed solution of Eq.(1.2), we obtain the linear equations (2.3) and have the eigenfunction corresponding to the eigenvalue λ_1 in the form

$$\phi[1,\lambda_1]_1 = d_1 e^{\mathscr{D}_1} + d_2 e^{\mathscr{D}_2}, \quad \phi[1,\lambda_1]_2 = d_3 e^{-\mathscr{D}_1} + d_2 e^{-\mathscr{D}_2}, \tag{4.3}$$

with

$$\begin{aligned}&\mathscr{D}_1 = \frac{1}{2} i(kx+\omega t) + \Gamma(x+\Delta t), \quad \mathscr{D}_2 = \frac{1}{2} i(kx+\omega t) - \Gamma(x+\Delta t),\\ &d_3 = \frac{d_2}{\gamma\sqrt{g_0}}\left(\frac{ik}{2} - \Gamma - \lambda_1\right), \quad \Gamma^2 = \left(\lambda_1 + \frac{k}{2}\right)^2 - \gamma^2, \ \ d_4 = \frac{d_1}{\gamma\sqrt{g_0}}\left(\frac{ik}{2} + \Gamma - \lambda_1\right),\\ &\Delta = k\lambda_1 - \epsilon k - 2i\epsilon\lambda_1 - i\frac{\epsilon k}{2} - \frac{\epsilon}{2} + i\gamma^2 - i\frac{\epsilon^2}{4} + (2i\epsilon - k)\left[\left(\lambda_1 + \frac{k}{2}\right)^2 - \gamma^2\right],\end{aligned} \tag{4.4}$$

where d_1 and d_2 are two arbitrary complex constants.

Substituting expressions (4.3) into formula (3.9), the one-soliton and multi-soliton solutions on the periodic background can be obtained with the iterative algorithm of the Darboux transformation. The bright one-soliton solutions are plotted in Figure 3, which shows four kinds of soliton profile structures with different wave numbers. Although solitons in Figures 3(c) and 3(d) hold larger wave numbers than those of Figures 3(a) and 3(b), they both can propagate stably for long distances by the results of numerical simulations of nonlinear pulses.

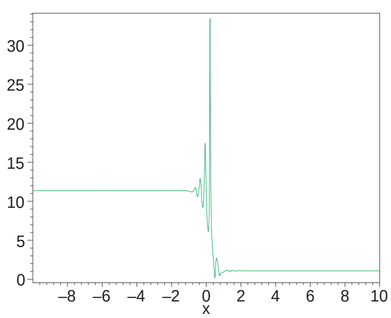
Proposition 4.1. *Assuming that* $(\phi[1,\lambda_1]_1, \phi[1,\lambda_1]_2)^T$ *be the solution of the linear eigenvalue problem* (2.3) *with eigenvalues* λ_1*, we obtain the bright two-soliton solutions of Eqs.*(1.2) *and* (2.2) *corresponding to eigenvalues* λ_2 *from formula* (3.9) *with* $N = 2$

$$\begin{aligned} \psi[2] &= u[2]e^{-i\frac{\epsilon x^2}{4}-\frac{\epsilon t}{2}}, \\ u[2] &= u(x,y)_0 + 2\sum_{i=1}^{2}\frac{(\lambda_i-\lambda_i^*)\phi[i,\lambda_i]_1\phi[i,\lambda_i]_2^*}{\phi[i,\lambda_i]_1\phi[i,\lambda_i]_1^*+\phi[i,\lambda_i]_2\phi[i,\lambda_i]_2^*}. \end{aligned} \tag{4.5}$$

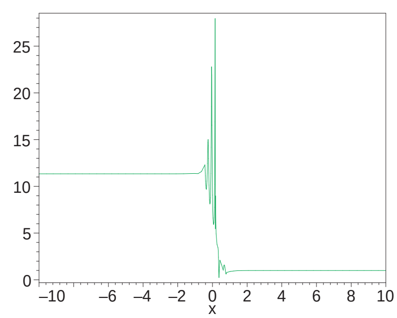
The solution of the linear eigenvalue problem (2.3) *with eigenvalues* λ_2 *is*

$$\begin{aligned} \phi[2,\lambda_2]_1 &= \frac{(\lambda_1+\lambda_1^*)\phi[1,\lambda_1]_1|\phi[1,\lambda_1]_2|^2-\lambda_1^*\phi[1,\lambda_1]_1\phi[1,\lambda_1]_2^*-\lambda_1\phi[1,\lambda_1]_1\phi[1,\lambda_1]_2^2}{\phi[1,\lambda_1]_1\phi[1,\lambda_1]_1^*+\phi[1,\lambda_1]_2\phi[1,\lambda_1]_2^*}, \\ \phi[2,\lambda_2]_2 &= \frac{(\lambda_1-\lambda_1^*)|\phi[1,\lambda_1]_1|^2\phi[1,\lambda_1]_2+\lambda_1^*|\phi[1,\lambda_1]_1|^2\phi[1,\lambda_1]_2^*-\lambda_1\phi[1,\lambda_1]_2^3}{\phi[1,\lambda_1]_1\phi[1,\lambda_1]_1^*+\phi[1,\lambda_1]_2\phi[1,\lambda_1]_2^*}. \end{aligned} \tag{4.6}$$

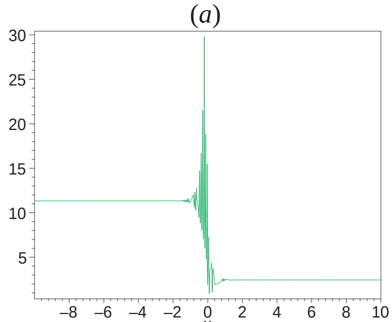
The bright two-soliton solutions are plotted in Figures 4. Figure 4 depicts the bright two-soliton periodic wave solutions in the three-dimensional space. From the interaction process in two sets of figures, it can be clearly seen that the interacting solitons like particles cross each other unaffectedly only by a phase shift, and their respective amplitudes and velocities are the same as those before collision. Further theoretical analysis for soliton solutions shows that the sign and value of real part of the eigenvalue determines the propagation direction of soltion and the amplitude, respectively.



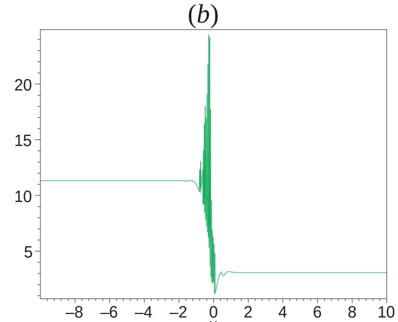
(a)



(b)



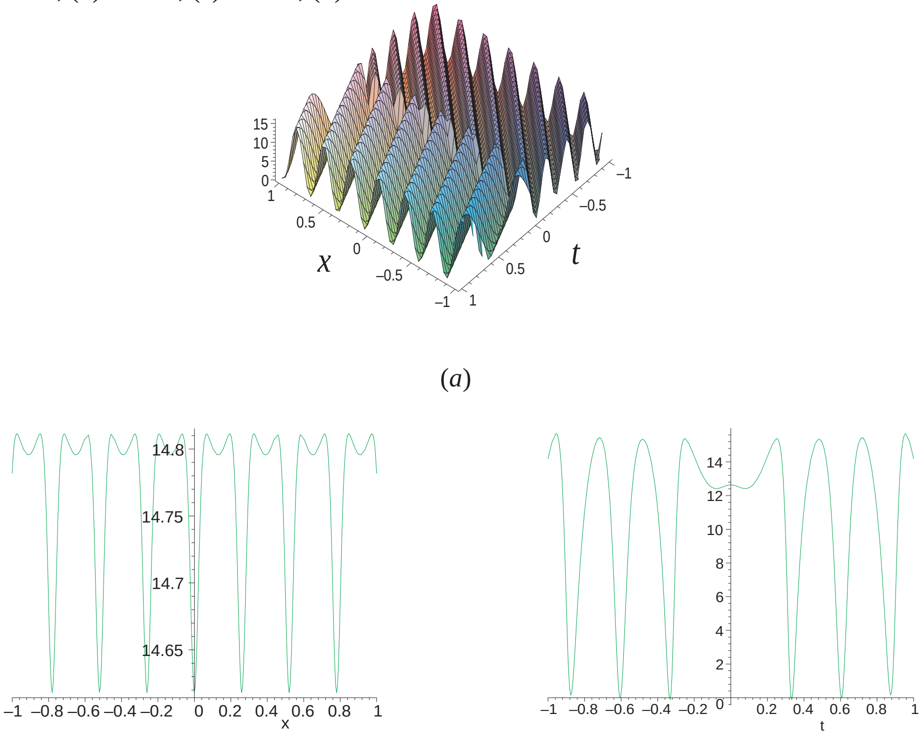
(c)



(d)

Fig. 3. (Color online) Four kinds of bright one-soliton solutions $|\psi(x,t)|^2$ of Eq.(1.2) with different wave

numbers k. The related physical quantities are $\gamma = 2, d_1 = 1, d_2 = 3, \lambda_1 = 5 - 2i, g_0 = 3, \epsilon = 0.001$ and wave numbers (a) $k = 1$, (b) $k = 10$, (c) $k = 50$, (d) $k = 100$.



(a)

(b) (c)

Fig. 4. (Color online) A asymmetric bright two-soliton solution $|\psi(x,t)|^2$ of Eq.(1.2) with parameters: $\gamma = 2, d_1 = 1, d_2 = 3, \lambda_1 = 10 + 2i, \lambda_2 = 10 - 2i, g_0 = 1, \epsilon = 0.001$ and wave numbers $k = 1$. This figure shows that the asymmetric bright two-soliton periodic wave is spatially periodic in two directions, but it need not be periodic in either the x or t directions. (a) Perspective view of the wave. (b) Wave propagation pattern of the wave along the x axis. (c) Wave propagation pattern of wave along the t axis.

5. Linear stability analysis

In this section, we analyze the modulational instability of the nonlinear plane waves. The phenomenon of modulational instability of nonlinear waves has a significant importance in the theory and experiment of nonlinear waves. The reason is that this phenomenon is quite relevant applications in many different branches of physics such as in condensate physics, plasma physics, hydrodynamics and nonlinear optical fibers [29]. Of particular interest for studies of stability of the solution of the NLS equation is that the envelope of a monochromatic plane wave propagates in nonlinear medium in the presence of microwave noise pulses.

As we well-known that the exact plane wave solution for the focusing NLS equation is to be linear unstable [30]. According to the method of investigating the linear stability of the NLS equation, we perform a linear stability analysis of the nonlinear plane waves solution for Eq.(1.2). The solution (4.1) is the plane wave with constant amplitude for Eq.(1.2). We investigate the solution with the perturbation in the form

$$\begin{aligned} &\psi(x,t) = u(x,t)e^{-i\frac{\epsilon x^2}{4} - \frac{\epsilon t}{2}}, \\ &u(x,t) = u_0(x,t)\left[1 + \varepsilon\tilde{u}(x,t)\right]. \end{aligned} \tag{5.1}$$

By substitution of (5.1) into Eq.(2.2), the linearized disturbance equation at $o(\varepsilon)$ becomes

$$i\tilde{u}_t + \tilde{u}_{xx} + i(2k - x\epsilon)\tilde{u}_x + 2\gamma^2 a(t)e^{\epsilon t}(\tilde{u}^* + 2\tilde{u}) + \epsilon(x-1)\left(\frac{\epsilon}{4}x + \frac{\epsilon}{4} + k\right)\tilde{u} = 0. \tag{5.2}$$

By virtue of the linearity of Eq.(5.2), the solution can be expressed as the following linear combination

$$\tilde{u}(x,t) = \mathscr{L}_+ e^{i\theta(x-\Theta t)} + \mathscr{L}_-^* e^{-i\theta(x-\Theta^* t)}. \tag{5.3}$$

We characterize this solution by the real disturbance wave number θ and the complex phase velocity Θ. Substituting (5.3) into (5.2) and collecting resonant terms, we have two linear homogeneous equations

$$\begin{aligned}
&\left[\theta\Theta - \theta^2 - \theta(2k - x\epsilon) + 4\gamma^2 a(t)e^{\epsilon t} + \epsilon(x-1)\left(\frac{\epsilon}{4}x + \frac{\epsilon}{4} + k\right)\right]\mathscr{L}_+ + 2\gamma^2 a(t)e^{\epsilon t}\mathscr{L}_- = 0,\\
&2\gamma^2 a(t)e^{\epsilon t}\mathscr{L}_+ + \left[-\theta\Theta - \theta^2 + \theta(2k - x\epsilon) + 4\gamma^2 a(t)e^{\epsilon t} + \epsilon(x-1)\left(\frac{\epsilon}{4}x + \frac{\epsilon}{4} + k\right)\right]\mathscr{L}_- = 0.
\end{aligned} \tag{5.4}$$

By using the determinant of the matrix of coefficients of linear Eqs.(5.4), the dispersion relation for linearized disturbance can be expressed as

$$\begin{aligned}
\Theta =& 2k - x\epsilon \pm \frac{1}{\theta}\sqrt{\left(\frac{\epsilon}{4}+k\right)^2\left[4\gamma^2 a(t)e^{\epsilon t} + \theta^2 - \epsilon\left(\frac{\epsilon}{4}+k\right)\right]^2 - 4\gamma^4 a(t)^2 e^{2\epsilon t}}\\
=& 2k - x\epsilon \pm \frac{1}{\theta}\sqrt{\left(\frac{\epsilon}{4}+k\right)^2\left[4\gamma^2 g_0 e^{2\epsilon t} + \theta^2 - \epsilon\left(\frac{\epsilon}{4}+k\right)\right]^2 - 4\gamma^4 g_0^2 e^{4\epsilon t}} \ \ (\text{with } \ \mathrm{a(t) = g_0 e^{\epsilon t}}),
\end{aligned} \tag{5.5}$$

where $0 < \epsilon \ll 1$ and γ, k are real constants.

By virtue of the above relation (5.5), if θ satisfies one of the inequalities as follows

$$\theta^2 \geq \frac{2\gamma^2 g_0 e^{2\epsilon t}}{\epsilon/4 + k} - 4\gamma^2 g_0 e^{2\epsilon t} + \epsilon\left(\frac{\epsilon}{4}+k\right) = \left(\frac{2}{\epsilon/4+k} - 4\right)\gamma^2 g_0 e^{2\epsilon t} + \epsilon\left(\frac{\epsilon}{4}+k\right), \tag{5.6}$$

$$\theta^2 \leq -\frac{2\gamma^2 g_0 e^{2\epsilon t}}{\epsilon/4 + k} + 4\gamma^2 g_0 e^{2\epsilon t} - \epsilon\left(\frac{\epsilon}{4}+k\right) = \left(4 - \frac{2}{\epsilon/4+k}\right)\gamma^2 g_0 e^{2\epsilon t} - \epsilon\left(\frac{\epsilon}{4}+k\right), \tag{5.7}$$

the frequency Θ is real at any value of the wavenumber θ, whereas Θ becomes complex.

Proposition 5.1 *(i) For Eq.*(5.6), *if* $\theta^2 \geq \epsilon\left(\frac{\epsilon}{4}+k\right)$ *and* $k \geq \frac{1}{4} > \frac{1}{2} - \frac{\epsilon}{4}$ $(0 < \epsilon \ll 1)$*, the frequency* Θ *is real at any value of the time variable t. Otherwise, there is the instability region of the modulational waves, and the disturbance will grow with time exponentially. (ii) For Eq.*(5.7)*, if* $0 \leq \theta^2 \leq -\epsilon\left(\frac{\epsilon}{4}+k\right)$ *and* $k < -\frac{\epsilon}{4}$ $(0 < \epsilon \ll 1)$*, the frequency* Θ *is real at any value of the time variable t. Otherwise, there is the instability region of the modulational waves, and the disturbance will grow with time exponentially.*

Compared with the analytical method, in the near future, we intend to study the modulational instability numerically for the 1D NLSE (1.2) since the linear stability analysis actually shows that this instability is possble.

6. Conclusions and discussions

In this paper, we investigate a general form of nonlinear Schrödinger equation with time-dependent nonlinearity, which describes conservation laws, bright matter wave solitons and modulational instability in Bose-Einstein condensates with the time-dependent interatomic interaction in an expulsive trapping potential. By virtue of the Lax pair, we have further confirmed the complete integrability of such a model by deriving an infinite number of conservation laws. We have adopted the Darboux transformation method to construct the multi-soliton solutions via algebraic iterative algorithm. We have discussed four kinds of bright one-soliton solutions $|\psi(x,t)|^2$ with different wave numbers k, and two-periodic bright solitons of Eq.(1.2) under the periodic background. Finally, we have analyzed the linear stability of nonlinear plane waves in the presence of small perturbation. In addition, it can be also applied to other spectral problems in the field of nonlinear partial differential equation appear in mathematical physics.

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References

[1] Gross E P, J. Math. Phys., 1963; 4: 1703944.

[2] Liang Z X, Zhang Z D and Liu W M, Phys. Rev. Lett., 2005; 94: 050402.

[3] Strecker K E, Partridge G B, Truscott A G and Hulet R G, Nature (London) 2002; 417: 150.

[4] Kevrekidis P G and Frantzeskakis D J; Mod. Phys. Lett. B, 2004; 18: 173.

[5] Brazhnyi V A and Konotop V V; Mod. Phys. Lett. B, 2004; 18: 627.

[6] Perez-Garcia V M, Konotop V V and Brazhnyi V A; Phys. Rev. Lett., 2004; 92: 220403.

[7] Gammal A, Frederico T and Tomio L; Phys. Rev. E, 1999; 60: 2421.

[8] Adhikari S K; Phys. Lett. A, 2000; 265: 91.

[9] Matveev V B and Salle M A, *Darboux Transformation and Solitons*, (Springer-Verlag, Berlin 1991).

[10] Gu C H, Hu H S and Zhou Z X, *Darboux Transformations in Soliton Theory and its Geometric Applications*, (Shanghai Sci Tech Publ., Shanghai 1999).

[11] Ablowitz M J and Segur H, *Solitons and the Inverse Scattering Transform*, (SIAM, Philadelphia 1981).

[12] Ablowitz M J, Kaup D J, Newell A C and Segur H, Stud. Appl. Math. 1974; 53: 249.

[13] Reyes E G, J. Math. Phys. 2000; 41: 2968.

[14] Li M, Tian B, Liu W J, Sun K, Jiang Y and Sun Z Y, Phys. Scr. 2010; 81: 045008.

[15] Tian S F and Zhang H Q, J. Nonlinear Math. Phys., 2010; 17: 491.

[16] Tian S F and Zhang H Q, Commun. Nonlinear Sci. Numer. Simul. 2011; 16: 173.

[17] Ma W X, Hu X B, Zhu S M and Wu Y T, J. Math. Phys., 1999; 40: 6071.

[18] Hu X B and Zhu Z N, J. Math. Phys., 1998; 39: 4766.

[19] Zhu Z N, Tam H W and Ding Q, Phys. Lett. A 2003; 310: 281.

[20] Lou S Y, Ruan H Y, Chen D F and Chen W Z, J. Phys. A, 1991; 24: 1455.

[21] Dai C Q and Wang Y Y, Phys. Scr. 2008; 78: 015013.

[22] Fan E G, Tam H W and Zhang Y F; Phys. Lett. A. 2006; 360: 99.

[23] Fan E G and Hon Y C; Phys. Lett. A 2002; 292: 335.

[24] Yan Z Y and Zhang H Q, Phys. Lett. A. 1999; 252: 291.

[25] Xia T C and Fan E G, J. Math. Phys., 2005; 46: 043510.

[26] Zha Q L and Li Z B, Phys. Lett. A. 2008; 372: 1422.

[27] Kivshar Y S, Physics Reports 1998; 298: 81.

[28] Gordon J P, Opt. Letts. 1983; 8: 596.

[29] Kamchatnov A M, *Nonlinear Periodic Waves and their Modulations* (World Sci., Hong Kong 2000)

[30] Wright O C, Nonlinearity, 1999; 12: 1277.

[31] Tian S F and Zhang H Q, Phys. Scr., 2010; 82: 015008.

[32] Tian S F, Wang Z and Zhang H Q, J. Math. Anal. Appl. 2010; 366: 646.